International Journal of Theoretical Physics, Vol. 39, No. 4, 2000

Unification of Bessel Functions of Different Orders

Mustapha Mekhfi[1]

Received July 2, 1999

We investigate the internal space of Bessel functions which is associated to the group Z of positive and negative integers defining their orders. As a result we propose and prove a new unifying formula (to be added to the huge literature on Bessel functions) generating Bessel functions of real orders out of integer order ones. The unifying formula is expected to be of great use in applied mathematics. Some applications of the formula are given for illustration

Early studies [e.g., 1, 2] proposed a unifying scheme for special functions showing, that these functions originated from the same structure. Their generating functions are representation "states" of the derivative and/or integral operators of arbitrary orders. More precisely, we have for the Bessel functions of concern here the "inner" structure

$$
\begin{aligned}
m &\to (-1)^m \partial_m, \; m \in Z \\
\partial_{|m|} &= \frac{\partial}{z\partial z}\frac{\partial}{z\partial z}\cdots\frac{\partial}{z\partial z} \\
\partial_{-|m|} &= \int z\,dz \int z\,dz \ldots \int z\,dz \qquad (1) \\
\partial_m \Phi(z, t) &= t^m\,\Phi(z), \; m \in Z
\end{aligned}
$$

where $\int dz$ is the "truncated" primitive, i.e., in defining the integral, we omit the constant of integration: $\int (df/dz)\,dz = f$. For polynomials such as Hermite and Laguerre polynomials, for instance, the generating functions only involve the realization of the set N of positive integers, with slight modifications of

[1] Laboratoire de Physique Mathematique, Es-senia 31100 Oran, Algeria and the Abdus Salam International Centre for Theoretical Physics, 34100 Trieste, Italy.

0020-7748/00/0400-1163$18.00/0

the derivative operators to account for the conventions used in defining these polynomials.

It is important to note that although this common structure only sets up the z dependence of the generating functions, it is the "dynamical" part of the scheme, so to say. The t dependence is simply set by imposing some given desired properties. For Bessel functions we require a "symmetry" between positive and negative indices, that is, $J_{-n} = (-1)^n J_n$, while for the polynomials it is the natural property of orthonormality that is invoked. In this paper we unify Bessel functions of integer orders with those of real orders as a complement to the unification of special functions initiated in the above-cited papers and subsequent papers. To our knowledge no such relation between integer and real orders is known to exist. Both functions are usually independently defined either by their own differential equations or by their expansions in the form of a series or integral. The mechanism underlying the unification we propose is of a different nature than the one described above, but makes use of the operators ∂_m, $m \in Z$, which serves to define Bessel functions of integer orders.

Let us first introduce the method by a simple example to see how the mechanism works to convert an integer into a real. Suppose we are given an abstract state $|n\rangle$, $n \in Z$, and a set of raising $\Pi(m)$, $m > 0$, and lowering $m < 0$, operators. Then it is easy to show, given that data, that the state $|n + \lambda\rangle$ is related to the state $|n\rangle$ through the following formula (provided the series involved converges, which is the case here):

$$|n + \lambda\rangle = \exp\left[-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}\right]|n\rangle \tag{2}$$

Fourier-transforming the $|n\rangle$ state as

$$|n\rangle = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, e^{in\theta}|\theta\rangle \tag{3}$$

where the $\Pi(m)$ operator acts on the $|\theta\rangle$ state by simple multiplication by the factor $e^{in\theta}$, We get

$$\begin{aligned} |n + \lambda\rangle &= \int_{-\pi}^{\pi} \exp\left[-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}\right] e^{in\theta}|\theta\rangle \frac{d\theta}{2\pi} \\ &= \int_{-\pi}^{\pi} e^{-i(n+\lambda)\theta}|\theta\rangle \frac{d\theta}{2\pi} \end{aligned} \tag{4}$$

In deriving the last line, use has been made of the known formula $\sum_{m=1}^{\infty} (-1)^m (\sin m\theta)/m = -\theta/2$, $-\pi < \theta < \pi$.

It is essential that the $|n\rangle$ states have the appropriate weights for the Fourier components, otherwise the series defining the transformation simply diverges. Reduced Bessel functions $\phi_n(z) = J_n(z)/z^n$ fit into the scheme. We indeed have a set of raising and lowering operators $\Pi(m) = (-1)^m\, \partial_m$, $m \in Z$ [3, 4]:

$$(-1)^m \frac{d^m}{(z\, dz)^m}\, \phi_n z = \phi_{n+m}(z), \qquad m \in N, \quad n \in Z \tag{5}$$

For negative m we just replace the derivative operators by the integral operators defined in (1). We then propose and prove the unifying formula

$$\frac{J_{n+\lambda}(z)}{z^{n+\lambda}} = \exp\left[-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}\right] \frac{J_n(z)}{z^n} \tag{6}$$

The operators act on the ϕ_n as

$$\Pi(m)\phi_n = \phi_{n+m} \tag{7}$$

To check the above formula, we could naively express the Bessel functions ϕ_{n+m} as entire series and then perform the sum over m. This is, however, a cumbersome procedure. A simple and illuminating way to proceed is to use the integral representation of ϕ_n, the "analog" of the $|n\rangle$ state Fourier decomposition,

$$\phi_n(z) = \frac{1}{2\pi i} \int_{l_0} \left(\frac{1}{2}\right)^n \tau^{-n-1} \exp\left(\tau - \frac{z^2}{4\tau}\right) d\tau \tag{8}$$

where τ is a complex variable and l_0 is a positively oriented closed path encircling the origin one time. Expanding the exponential in (6) as

$$\begin{aligned}\phi_{n+\lambda}(z) &= \exp\left[-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}\right]\phi_n(z) \\ &= \sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} \left[\sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}\right]^p \phi_n(z)\end{aligned} \tag{9}$$

and using the propery that $\Pi(m_1) \ldots \Pi(m_p) = \Pi(m_1 + \ldots m_p)$, we find that the term of order λ^p takes the form

$$\frac{(-\lambda)^p}{p!} \sum_{m_1} \ldots \sum_{m_p} \frac{(-1)^{m_1 + \ldots m_p}}{m_1 \cdots m_p}\, \phi_{m_1 + \cdots m_p} \tag{10}$$

Inserting the ϕ_n integral representation into (10), we get

$$\frac{1}{2\pi i}\frac{(-\lambda)^p}{p!}\int_{l_0}\sum_{m_1}\cdots\sum_{m_p}\frac{(-1)^{m_1+\ldots m_p}}{m_1\cdots m_p}\left(\frac{1}{2}\right)^{m_1+\ldots m_p}$$

$$\times\ \tau^{-(m_1+\ldots m_p+n)-1}\exp\left(\tau-\frac{z^2}{4\tau}\right)d\tau$$

$$=\frac{1}{2\pi i}\frac{-\lambda^p}{p!}\int_{l_0}\left(\frac{1}{2}\right)^n\tau^{-n-1}\left[\sum_m(-1)^m\frac{(2\tau)^{-m}}{m}\right]^p\exp\left(\tau-\frac{z^2}{4\tau}\right)\tag{11}$$

The relevant term to sum is

$$\sum_{m\in Z/(0)}(-1)^m\frac{(2\tau)^{-m}}{m}\tag{12}$$

It is to be noted that the above series diverges on the whole complex plane except on the circle centering the origin and with half unity radius, on which it converges uniformally. It is only on that circle that we have the right to commute the signs Σ and $\int$ when we insert (8) into (10). Now the miracle happens. To make the series converge, we just deform the path l_0 down to the above circle, as the integrand in (8) has only essential singularities at $\tau = 0$ and $\tau = \infty$. To compute the relevant sum, put $2\tau = e^{i\theta}$, the series then converges to

$$\begin{aligned}\sum_{m\in Z/(0)}(-1)^m\frac{(2\tau)^{-m}}{m} &= -2i\sum_{m=1}^{m=\infty}(-1)^m\frac{\sin m\theta}{m}\\ &= i\theta\quad -\pi<\theta<\pi \\ &= \ln 2\tau\end{aligned}\tag{13}$$

where the branch cut of the logarithm is taken along the negative real axis as θ ranges from $-\pi$ to π. Inserting the result into (11), the latter becomes

$$\frac{1}{2\pi i}\frac{(-\lambda)^p}{p!}\int_{l}\left(\frac{1}{2}\right)^n\tau^{-(n+\lambda)-1}(\ln 2\tau)^p\exp\left(\tau-\frac{z^2}{4\tau}\right)d\tau\tag{14}$$

where the new, positively oriented path l now goes around the cut lying on the negative real axis. Expression (14) is, however, nothing but the λ^p-order term of the real order Bessel function $\phi_{n+\lambda}$ when it is written in terms of its integral representation,

$$\phi_{n+\lambda}(z)=\frac{1}{2\pi i}\int_{l_0}\left(\frac{1}{2}\right)^{n+\lambda}\tau^{-(n+\lambda)-1}\exp\left(\tau-\frac{z^2}{4\tau}\right)d\tau\tag{15}$$

To see this, rewrite $(2\tau)^{-\lambda} = \exp(-\lambda\ln 2\tau)$ and take the λ^p order, which is $(1/p!)(-\lambda)^p(\ln 2\tau)^p$. We thus proved the proposed unifying formula.

Now we propose two applications of this formula for illustration. For first immediate application, consider that Bessel functions enjoy the essential property

$$(-1)^m \frac{d^m}{(zdz)^m} \phi_n(z) = \phi_{n+m}(z), \qquad n \in Z \tag{16}$$

This formula is easily generalized to $\phi_{n+\lambda}$ with λ real since the operator $d^m/(zdz)^m$ appearing in (16) commutes with the operator defining $\phi_{n+\lambda}$ in (6). The second application is the computation of the generating function of the reduced real order Bessel function assuming that we know the generating function of the reduced integer order Bessel function. That is we want indirectly to compute

$$\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z)t^n \tag{17}$$

where t is a complex parameter. To perform the sum, we use the unifying formula.

$$\begin{aligned}\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z)t^n &= \sum_{n=-\infty}^{n=\infty} \exp\left[-\lambda \sum_{m\in Z/(0)} (-1)^m \frac{\Pi(m)}{m}\right]\phi_n t^n \\ &= \exp\left[-\lambda \sum_{m\in Z/(0)} (-1)^m \frac{\Pi(m)}{m}\right]\Phi(z, t)\end{aligned} \tag{18}$$

where $\Phi(z, t)$ is the generating function for Bessel functions of integer orders. That is, $\Phi(z, t) = \Sigma_{n=-\infty}^{\infty} \phi_n(z)t^n$. The action of the $\Pi(m)$ operators is simply.

$$\Pi(m)\Phi = t^{-m}\Phi \tag{19}$$

Then the third line in (18) involves the sum

$$\sum_{m\in Z/(0)} (-1)^m \frac{t^{-m}}{m} \tag{20}$$

This series is only convergent on the circle of radius unity, so that we put $t = e^{i\theta}$ and find,

$$\begin{aligned}\sum_{m\in Z/(0)} (-1)^m \frac{e^{-im\theta}}{m} &= i\theta - \pi < \theta < \pi \\ &= \ln t\end{aligned} \tag{21}$$

Putting this into (18) we get the end result that the generating function for $J_{n+\lambda}/z^{n+\lambda}$ as defined above is $t^{-\lambda}$ times the generating function for Bessel functions of integer orders. Our formula shows that this is true on the cirle of radius unity except on the the point -1,

$$\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z)t^n = t^{-\lambda} \sum_{n=-\infty}^{n=\infty} \phi_n(z)t^n = t^{-\lambda} \exp\left(\frac{t}{2} - \frac{z^2}{2t}\right) \tag{22}$$

Note that our formula only predicts the result on the circle. If we assume in addition that the series is regular on the complex plane except on the branch cut on the negative axis, then the result we found is valid outside the circle and covers the whole domain of regularity. The unifying formula will show its power in other interesting applications which are under active investigations.

Let us for comparison recompute the series using a direct method. We will use the integral representation of the reduced Bessel function, otherwise the computation is simply awful:

$$\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z)t^n = \frac{t^{-\lambda}}{2\pi i} \sum_{n=-\infty}^{n=\infty} \int_l \left(\frac{t}{2\tau}\right)^{(n+\lambda)} \exp\left(\tau - \frac{z^2}{4\tau}\right) \frac{d\tau}{\tau} \tag{23}$$

Making the change of variable $t/(2\tau) = e^{i\theta}$, we rewrite the above expression as

$$\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z)t^n = \frac{t^{-\lambda}}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{n=\infty} e^{in\theta}\, e^{i\lambda\theta} \exp\left(\frac{t}{2}\, e^{-i\theta} - \frac{z^2}{2t}\, e^{i\theta}\right) d\theta \tag{24}$$

knowing that the involved series converges to the sum of Dirac distributions,

$$\sum_{n=-\infty}^{n=\infty} e^{in\theta} = \sum_{p=-\infty}^{p=\infty} 2\pi\delta(\theta - 2\pi p) \tag{25}$$

we get after insertion of the above result

$$\begin{aligned}
&\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z)t^n \\
&\quad = t^{-\lambda} \int_{-\pi}^{\pi} \sum_{p=-\infty}^{p=\infty} \delta(\theta - 2\pi p)e^{i\lambda\theta} \exp\left(\frac{t}{2}\, e^{-i\theta} - \frac{z^2}{2t}\, e^{i\theta}\right) d\theta \\
&\quad = t^{-\lambda} \exp\left(\frac{t}{2} - \frac{z^2}{2t}\right)
\end{aligned} \tag{26}$$

Note that only the $p = 0$ term in the above sum contributes since the path of integration encircles the origin only one time. Here again we get the same result with the advantage, however, that the result is valid on the whole complex t plane except the cut. This is because we can always deform the τ contour so as to make $|t/(2\tau)| = 1$ and hence to sum the series.

We conclude with the following worth remarks. First, we have shown that Bessel functions of real orders can be obtained from integer order ones through the mechanism we described above, and from this point of view integer order Bessel functions are more "fundamental". Second, there are some well known functions in mathematical physics which are simply constructed from various combinations of the linearly independent $J_p(z)$ and $J_{-p}(z)$, where p is real. These are Hankel $H_p^1(z)$, $H_p^2(z)$ and Neumann functions $N_p(z)$, which are expressed as

$$\begin{aligned} N_p(z) &= \frac{J_p(z)\cos p\pi - J_{-p}}{\sin p\pi} \\ H^1(z) &= J_p(z) + iN_p(z) \\ H^2(z) &= J_p(z) - iN_p(z) \end{aligned} \tag{27}$$

These functions are also connected with the above unification, but to give the formula which relates integer orders to real ones is not a straightforward matter, although they are simply linear combinations of Bessel functions. The reason is that the unifying formula acts on $J_n(z)/z^n$ and not directly on $J_n(z)$ and therefore we cannot naively apply it to the defining formulas (27) with p integer. We managed, however, to unify Neumann functions of different orders using our unifying formula, but following an indirect way. The result will appear in a forthcoming paper.

ACKNOWLEDGMENT

I would like to thank the head of the high energy section at the Abdus Salam International Centre for Theoretical Physics, Dr. S. Randjbar-Daemi, for continuous help and encouragment during my stay as guest scientist at the Centre.

REFERENCES

1. C. Truesdell, *An Essay Toward a Unified Theory of Special Functions*, Princeton University Press, Princeton, New Jersey, 1948.
2. L. Infeld and T. Hull, The factorization method, *Rev. Mod. Phys.* **23** (1951) 21–68.
3. G. N. Watson, *A Treatise on Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge.
4. V. I. Smirnov, *A Course on Higher Mathematics*, Vol. III, Part 2, Pergamon Press, Oxford, 1964.